2.4 Rates of Change and Tangent Lines

What you will learn about . . .

- Average Rates of Change
- Tangent to a Curve
- Slope of a Curve
- Normal to a Curve
- Speed Revisited

and why . . .

The tangent line determines the direction of a body's motion at every point along its path.

Average Rates of Change

We encounter average rates of change in such forms as average speed (in miles per hour), growth rates of populations (in percent per year), and average monthly rainfall (in inches per month). The **average rate of change** of a quantity over a period of time is the amount of change divided by the time it takes. In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

EXAMPLE 1 Finding Average Rate of Change

Find the average rate of change of $f(x) = x^3 - x$ over the interval $[1, 3]$.

SOLUTION

Since $f(1) = 0$ and $f(3) = 24$, the average rate of change over the interval $[1, 3]$ is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{24 - 0}{2} = 12.$$

Now Try Exercise 1.

Experimental biologists often want to know the rates at which populations grow under controlled laboratory conditions. Figure 2.27 shows how the number of fruit flies *(Drosophila)* grew in a controlled 50-day experiment. The graph was made by counting flies at regular intervals, plotting a point for each count, and drawing a smooth curve through the plotted points.

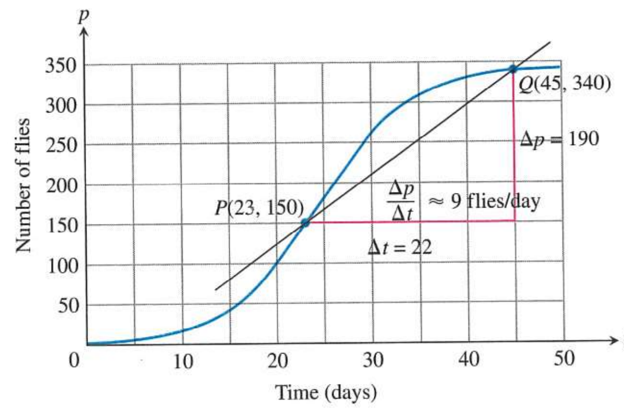


Figure 2.27 Growth of a fruit fly population in a controlled experiment.
Source: Elements of Mathematical Biology. (Example 2)

Secant to a Curve

A line through two points on a curve is a **secant to the curve.**

EXAMPLE 2 Growing *Drosophila* in a Laboratory

Use the points $P(23, 150)$ and $Q(45, 340)$ in Figure 2.27 to compute the average rate of change and the slope of the secant line PQ.

SOLUTION

There were 150 flies on day 23 and 340 flies on day 45. This gives an increase of $340 - 150 = 190$ flies in $45 - 23 = 22$ days.

The average rate of change in the population p from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day},$$

or about 9 flies per day.

continued

Marjorie Lee Browne (1914–1979)

When Marjorie Browne graduated from the University of Michigan in 1949, she was one of the first two African American women to be awarded a Ph.D. in Mathematics. Browne went on to become chairperson of the mathematics department at North Carolina Central University, and succeeded in obtaining grants for retraining high school mathematics teachers.

This average rate of change is also the slope of the secant line through the two points P and Q on the population curve. We can calculate the slope of the secant PQ from the coordinates of P and Q.

$$\textit{Secant slope:}\quad \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}$$

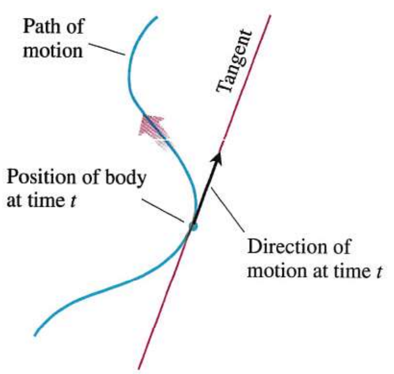
Now Try Exercise 7.

As suggested by Example 2, *we can always think of an average rate of change as the slope of a secant line.*

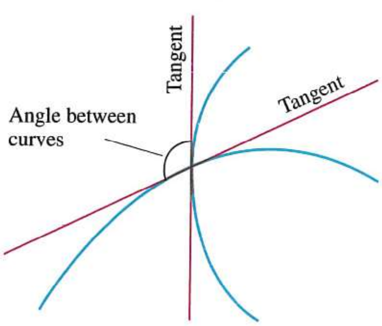
In addition to knowing the average rate at which the population grew from day 23 to day 45, we may also want to know how fast the population was growing on day 23 itself. To find out, we can watch the slope of the secant PQ change as we back Q along the curve toward P. The results for four positions of Q are shown in Figure 2.28.

Why Find Tangents to Curves?

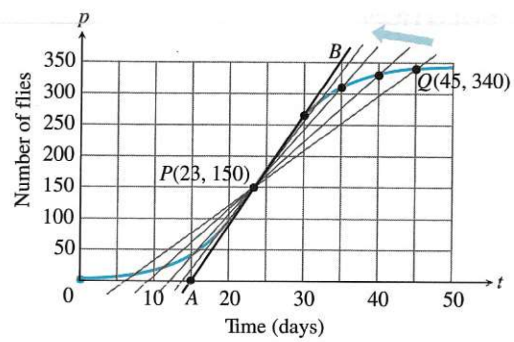
In mechanics, the tangent determines the direction of a body's motion at every point along its path.



In geometry, the tangents to two curves at a point of intersection determine the angle at which the curves intersect.



In optics, the tangent determines the angle at which a ray of light enters a curved lens (more about this in Section 4.2). The problem of how to find a tangent to a curve became the dominant mathematical problem of the early 17th century, and it is hard to overestimate how badly the scientists of the day wanted to know the answer. Descartes went so far as to say that the problem was the most useful and most general problem not only that he knew but that he had any desire to know.



(a)

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	(340 – 150)/(45 – 23) ≈ 8.6
(40, 330)	(330 – 150)/(40 – 23) ≈ 10.6
(35, 310)	(310 – 150)/(35 – 23) ≈ 13.3
(30, 265)	(265 – 150)/(30 – 23) ≈ 16.4

(b)

Figure 2.28 (a) Four secants to the fruit fly graph of Figure 2.27, through the point $P(23, 150)$. (b) The slopes of the four secants.

In terms of geometry, what we see as Q approaches P along the curve is this: The secant PQ approaches the tangent line AB that we drew by eye at P. This means that within the limitations of our drawing, the slopes of the secants approach the slope of the tangent, which we calculate from the coordinates of A and B to be

$$\frac{350 - 0}{35 - 15} = 17.5 \text{ flies/day.}$$

In terms of population, what we see as Q approaches P is this: The average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at P (17.5 flies per day). The slope of the tangent line is therefore the number we take as the rate at which the fly population was growing on day $t = 23$.

Tangent to a Curve

The moral of the fruit fly story would seem to be that we should define the rate at which the value of the function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$. But how are we to define the tangent line at an arbitrary point P on the curve and find its slope from the formula $y = f(x)$? The problem here is that we know only one point. Our usual definition of slope requires two points.

The solution that mathematician Pierre Fermat found in 1629 proved to be one of that century's major contributions to calculus. We still use his method of defining tangents to produce formulas for slopes of curves and rates of change:

1. We start with what we can calculate, namely, the slope of a secant through P and a point Q nearby on the curve.

2. We find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.

3. We define the *slope of the curve* at P to be this number and define the *tangent to the curve* at P to be the line through P with this slope.

EXAMPLE 3 Finding Slope and Tangent Line

Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

SOLUTION

We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$ on the curve (Figure 2.29).

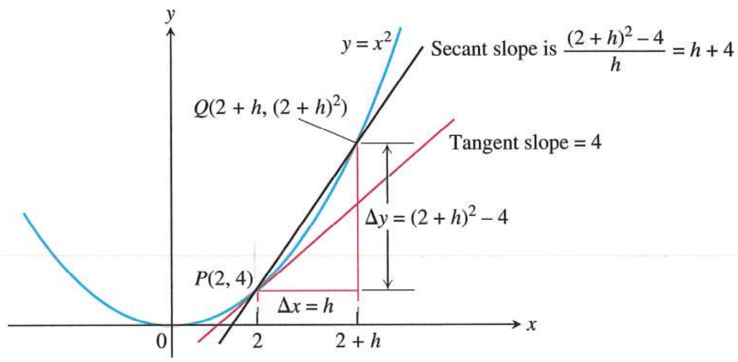


Figure 2.29 The slope of the tangent to the parabola $y = x^2$ at $P(2, 4)$ is 4.

Pierre de Fermat (1601–1665)

The dynamic approach to tangency, invented by Fermat in 1629, proved to be one of the 17th century's major contributions to calculus. Fermat, a skilled linguist and one of his century's greatest mathematicians, tended to confine his writing to professional correspondence and to papers written for personal friends. He rarely wrote completed descriptions of his work, even for his personal use. His name slipped into relative obscurity until the late 1800s, and it was only from a four-volume edition of his works published at the beginning of this century that the true importance of his many achievements became clear.

We then write an expression for the slope of the secant line and find the limiting value of this slope as Q approaches P along the curve.

$$\begin{aligned}\text{Secant slope} &= \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 4}{h} \\ &= \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4\end{aligned}$$

The limit of the secant slope as Q approaches P along the curve is

$$\lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (h + 4) = 4.$$

Thus, the slope of the parabola at P is 4.

The tangent to the parabola at P is the line through $P(2, 4)$ with slope $m = 4$.

$$\begin{aligned} y - 4 &= 4(x - 2) \\ y &= 4x - 8 + 4 \\ y &= 4x - 4 \end{aligned}$$

Now Try Exercise 11 (a, b).

Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$ we use the same dynamic procedure. We calculate the slope of the secant line through P and a point $Q(a + h, f(a + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 2.30). If the limit exists, it is the slope of the curve at P and we define the tangent at P to be the line through P having this slope.

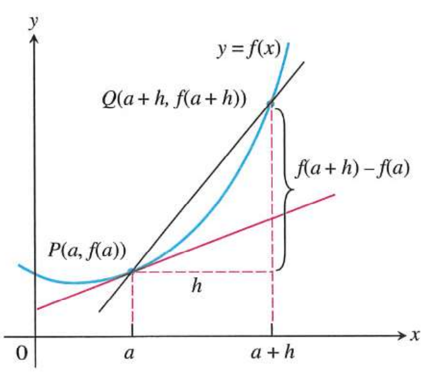


Figure 2.30 The tangent slope is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

DEFINITION Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

EXAMPLE 4 Exploring Slope and Tangent

Let $f(x) = 1/x$.

(a) Find the slope of the curve at $x = a$.

(b) Where does the slope equal $-1/4$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ for different values of a?

SOLUTION

(a) The slope at $x = a$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \cdot \frac{-h}{ha(a+h)}$$

$$= \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

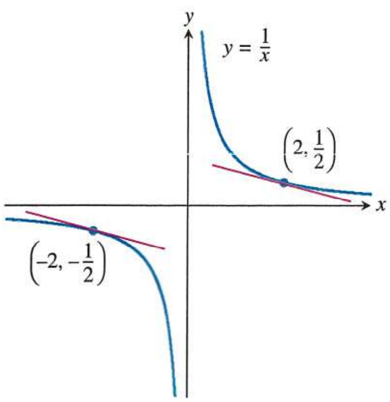


Figure 2.31 The two tangent lines to $y = 1/x$ having slope $-1/4$. (Example 4)

(b) The slope will be $-1/4$ if

$$-\frac{1}{a^2} = -\frac{1}{4}$$

$$a^2 = 4 \qquad \text{Multiply by } -4a^2.$$

$$a = \pm 2.$$

The curve has the slope $-1/4$ at the two points $(2, 1/2)$ and $(-2, -1/2)$ (Figure 2.31).

(c) The slope $-1/a^2$ is always negative. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep. We see this again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent becomes increasingly horizontal. ***Now Try Exercise 19.***

All of These Are the Same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

The expression

$$\frac{f(a+h) - f(a)}{h}$$

is the **difference quotient of *f* at *a*.** Suppose the difference quotient has a limit as h approaches zero. If we interpret the difference quotient as a secant slope, the limit is the slope of both the curve and the tangent to the curve at the point $x = a$. If we interpret the difference quotient as an average rate of change, the limit is the function's rate of change with respect to x at the point $x = a$. This limit is one of the two most important mathematical objects considered in calculus. We will begin a thorough study of it in Chapter 3.

An Alternate Form

In Chapter 3, we will introduce the expression

$$\frac{f(x) - f(a)}{x - a}$$

as an important and useful alternate form of the **difference quotient of *f* at *a*.** (See Exercise 55.)

About the Word *Normal*

When analytic geometry was developed in the 17th century, European scientists still wrote about their work and ideas in Latin, the one language that all educated Europeans could read and understand. The Latin word *normalis,* which scholars used for *perpendicular,* became *normal* when they discussed geometry in English.

Normal to a Curve

The **normal line** to a curve at a point is the line perpendicular to the tangent at that point.

EXAMPLE 5 Finding a Normal Line

Write an equation for the normal to the curve $f(x) = 4 - x^2$ at $x = 1$.

SOLUTION

The slope of the tangent to the curve at $x = 1$ is

$$\begin{aligned}\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0} \frac{4 - (1+h)^2 - 3}{h} \\ &= \lim_{h \to 0} \frac{4 - 1 - 2h - h^2 - 3}{h} \\ &= \lim_{h \to 0} \frac{-h(2+h)}{h} = -2.\end{aligned}$$

Thus, the slope of the normal is $1/2$, the negative reciprocal of -2. The normal to the curve at $(1, f(1)) = (1, 3)$ is the line through $(1, 3)$ with slope $m = 1/2$.

$$\begin{aligned}y - 3 &= \frac{1}{2}(x - 1) \\ y &= \frac{1}{2}x - \frac{1}{2} + 3 \\ y &= \frac{1}{2}x + \frac{5}{2}\end{aligned}$$

You can support this result by drawing the graphs in a square viewing window.

Now Try Exercise 11 (c, d).

Speed Revisited

The function $y = 16t^2$ that gave the distance fallen by the rock in Example 1, Section 2.1, was the rock's *position function.* A body's average speed along a coordinate axis (here, the y-axis) for a given period of time is the average rate of change of its *position* $y = f(t)$. Its *instantaneous speed* at any time t is the **instantaneous rate of change** of position with respect to time at time t, or

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}.$$

Particle Motion

We only have considered objects moving in one direction in this chapter. In Chapter 3, we will deal with more complicated motion.

We saw in Example 1, Section 2.1, that the rock's instantaneous speed at $t = 2$ sec was 64 ft/sec.

EXAMPLE 6 Finding Instantaneous Rate of Change

Find

$$\lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

for the function $f(t) = 2t^2 - 1$ at $t = 2$. Interpret the answer if $f(t)$ represents a position function in feet of an object at time t seconds.

continued

SOLUTION

$$\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} = \lim_{h\to 0}\frac{2(2+h)^2-1-(2\cdot 2^2-1)}{h}$$
$$= \lim_{h\to 0}\frac{8+8h+2h^2-1-7}{h}$$
$$= \lim_{h\to 0}\frac{8h+2h^2}{h}$$
$$= \lim_{h\to 0}(8+2h) = 8$$

The instantaneous rate of change of the object is 8 ft/sec.

Now Try Exercise 23.

EXAMPLE 7 Investigating Free Fall

Find the speed of the falling rock in Example 1, Section 2.1, at $t = 1$ sec.

SOLUTION

The position function of the rock is $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ sec was

$$\frac{f(1+h)-f(1)}{h} = \frac{16(1+h)^2-16(1)^2}{h} = \frac{16(h^2+2h)}{h} = 16(h+2).$$

The rock's speed at the instant $t = 1$ was

$$\lim_{h\to 0} 16(h+2) = 32 \text{ ft/sec}.$$

Now Try Exercise 31.

Quick Review 2.4 *(For help, go to Section 1.1.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B.

1. $A(-5, 2)$, $B(3, 5)$

2. $A(1, 3)$, $B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3)$, $(5, -1)$

4. $(-3, -1)$, $(3, 3)$

In Exercises 5–9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= 3/2$

6. through $(1, 6)$ and $(4, -1)$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $5/3$?

Section 2.4 Exercises

In Exercises 1–6, find the average rate of change of the function over each interval.

1. $f(x) = x^3 + 1$
(a) $[2, 3]$ **(b)** $[-1, 1]$

2. $f(x) = \sqrt{4x+1}$
(a) $[0, 2]$ **(b)** $[10, 12]$

3. $f(x) = e^x$
(a) $[-2, 0]$ **(b)** $[1, 3]$

4. $f(x) = \ln x$
(a) $[1, 4]$ **(b)** $[100, 103]$

5. $f(x) = \cot x$
(a) $[\pi/4, 3\pi/4]$ **(b)** $[\pi/6, \pi/2]$

6. $f(x) = 2 + \cos x$
(a) $[0, \pi]$ **(b)** $[-\pi, \pi]$