Lecture 1, Chap 2.1 Rates of Change and tangents to Curves

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Let AVR = Average Rate of Change

Remember slope $m = \frac{y_2 - y_1}{x_2 - x_1}$

#2) $g(x) = x^2 - 2x$

a) $[1, 3]$

$$g(1) = 1^2 - 2 \cdot 1 = 1 - 2 = -1$$

$$g(1) = -1$$

$$g(3) = 3^2 - 2 \cdot 3 = 9 - 6 = 3$$

$$g(3) = 3$$

$$AVR = \frac{g(1) - g(3)}{3 - 1}$$

$$AVR = \frac{-1-3}{3-1}$$

$$AVR = \frac{-4}{2} = -2$$

b. $[-2, 4]$

$$g(-2) = (-2)^2 - 2(-2)$$

$= 4 + 4$

$g(-2) = 8$

$g(4) = 4^2 - 2(4)$

$$= 16 - 8$$

$$g(4) = 8$$

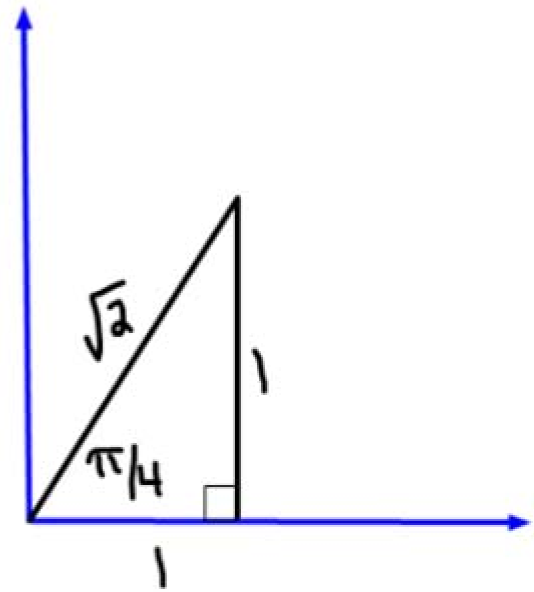
$$AVR = \frac{g(-2) - g(4)}{4 - 2}$$

$$AVR = \frac{8-8}{2}$$

$$AVR = \frac{0}{2} = 0$$

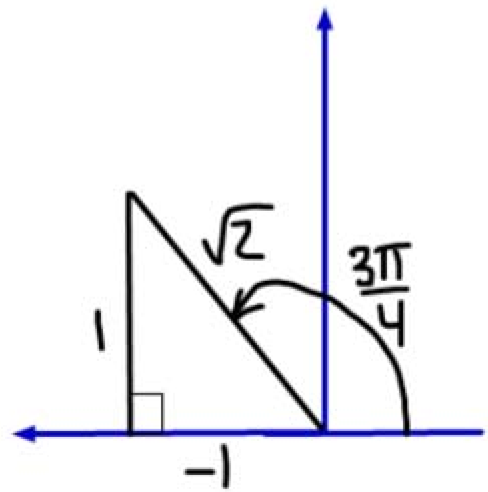
#3) $h(t) = \cot t$

a. $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$



$$h\left(\frac{\pi}{4}\right) = \cot\left(\frac{\pi}{4}\right) = 1$$

$$h\left(\frac{3\pi}{4}\right) = \cot\left(\frac{3\pi}{4}\right) = -1$$



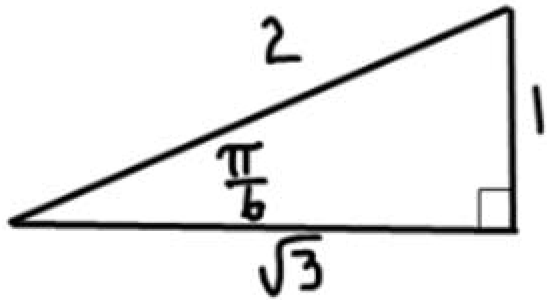
$$AVR = \frac{\cot\left(\frac{3\pi}{4}\right) - \cot\left(\frac{\pi}{4}\right)}{\frac{3\pi}{4} - \frac{\pi}{2}}$$

$$AVR = \frac{-1-1}{\left(\frac{2\pi}{4}\right)}$$

$$AVR = -2\left(\frac{2}{\pi}\right) = \frac{-4}{\pi}$$

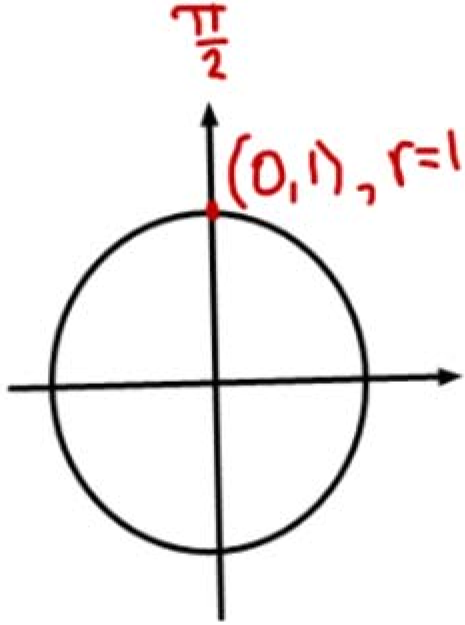
b. $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

$$h\left(\frac{\pi}{6}\right) = \cot\left(\frac{\pi}{6}\right)$$



$$h\left(\frac{\pi}{6}\right) = \sqrt{3}$$

$$h\left(\frac{\pi}{2}\right) = \cot\left(\frac{\pi}{2}\right)$$



$$h\left(\frac{\pi}{2}\right) = \frac{0}{1} = 0$$

$$AVR = \frac{h\left(\frac{\pi}{6}\right) - h\left(\frac{\pi}{2}\right)}{\frac{\pi}{6} - \frac{\pi}{2}}$$

$$AVR = \frac{\sqrt{3} - 0}{\frac{\pi}{6} - \frac{3\pi}{6}}$$

$$AVR = \frac{\sqrt{3}}{-\frac{2\pi}{6}}$$

$$AVR = \frac{\sqrt{3}}{\left(-\frac{\pi}{6}\right)}$$

$$AVR = \sqrt{3}\left(-\frac{6}{\pi}\right) = \frac{-6\sqrt{3}}{\pi}$$

#4) $g(t) = 2 + \cos t$

a. $[0, \pi]$

$$g(0) = 2 + \cos 0$$

$$g(0) = 2 + 1$$

$$g(0) = 3$$

$$g(\pi) = 2 + \cos(\pi)$$

$$g(\pi) = 2 - 1$$

$$g(\pi) = 1$$

$$AVR = \frac{g(\pi) - g(0)}{\pi - 0}$$

$$AVR = \frac{1-3}{\pi}$$

$AVR = \frac{-2}{\pi}$

b. $[-\pi, \pi]$

$g(-\pi) = 2 + \cos(-\pi)$

$g(-\pi) = 2 - 1$

$$g(-\pi) = 1$$

$$g(\pi) = 2 + \cos(\pi)$$

$$g(\pi) = 2 - 1$$

$$g(\pi) = 1$$

$$AVR = \frac{g(\pi) - g(-\pi)}{\pi - (-\pi)}$$

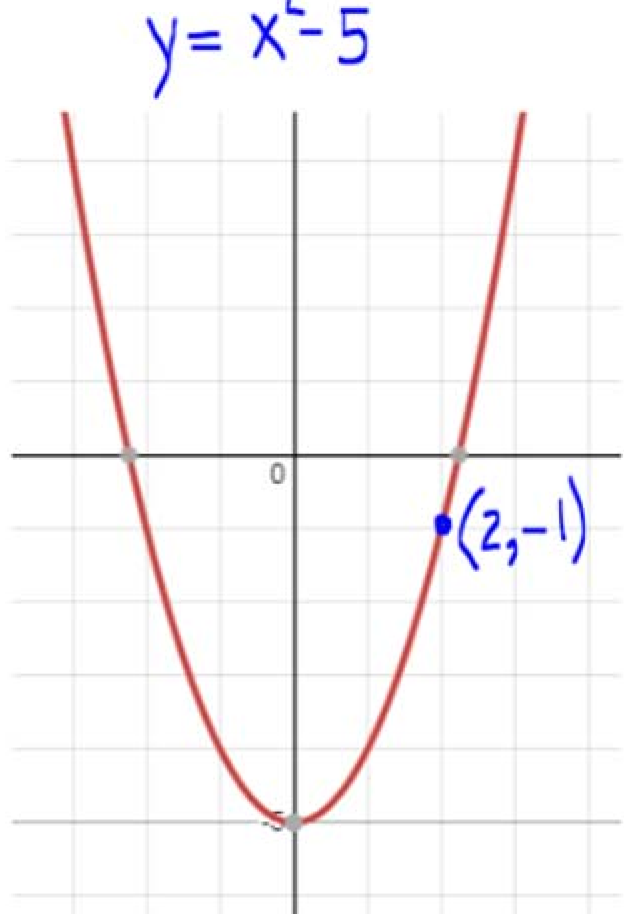
$$AVR = \frac{1-1}{\pi+\pi}$$

$$AVR = \frac{0}{2\pi} = 0$$

#7) $y = x^2 - 5 \quad P(2,-1)$

For $h > 0$, Let $h = 0.001$

For $h < 0$, Let $h = -0.001$



$x_1 = 2 \qquad y_1(2) = -1$

$x_2 = 2 + h \qquad y_2(2+h) = (2+h)^2 - 5$

$$\text{secant line slope} = \frac{y_2(2+h) - y_1(2)}{x_2 - x_1}$$

$$= \frac{\left[(2+h)^2 - 5\right] - (-1)}{(2+h) - 2}$$

$$= \frac{(2+h)^2 - 5 + 1}{h}$$

$$= \frac{4 + 4h + h^2 - 4}{h}$$

$$= \frac{4h + h^2}{h}$$

$$= \frac{h(4+h)}{h}$$

$$= 4+h$$

If $h>0$, then Q lies above to the right of P.

If $h<0$, then Q lies to the left of P.

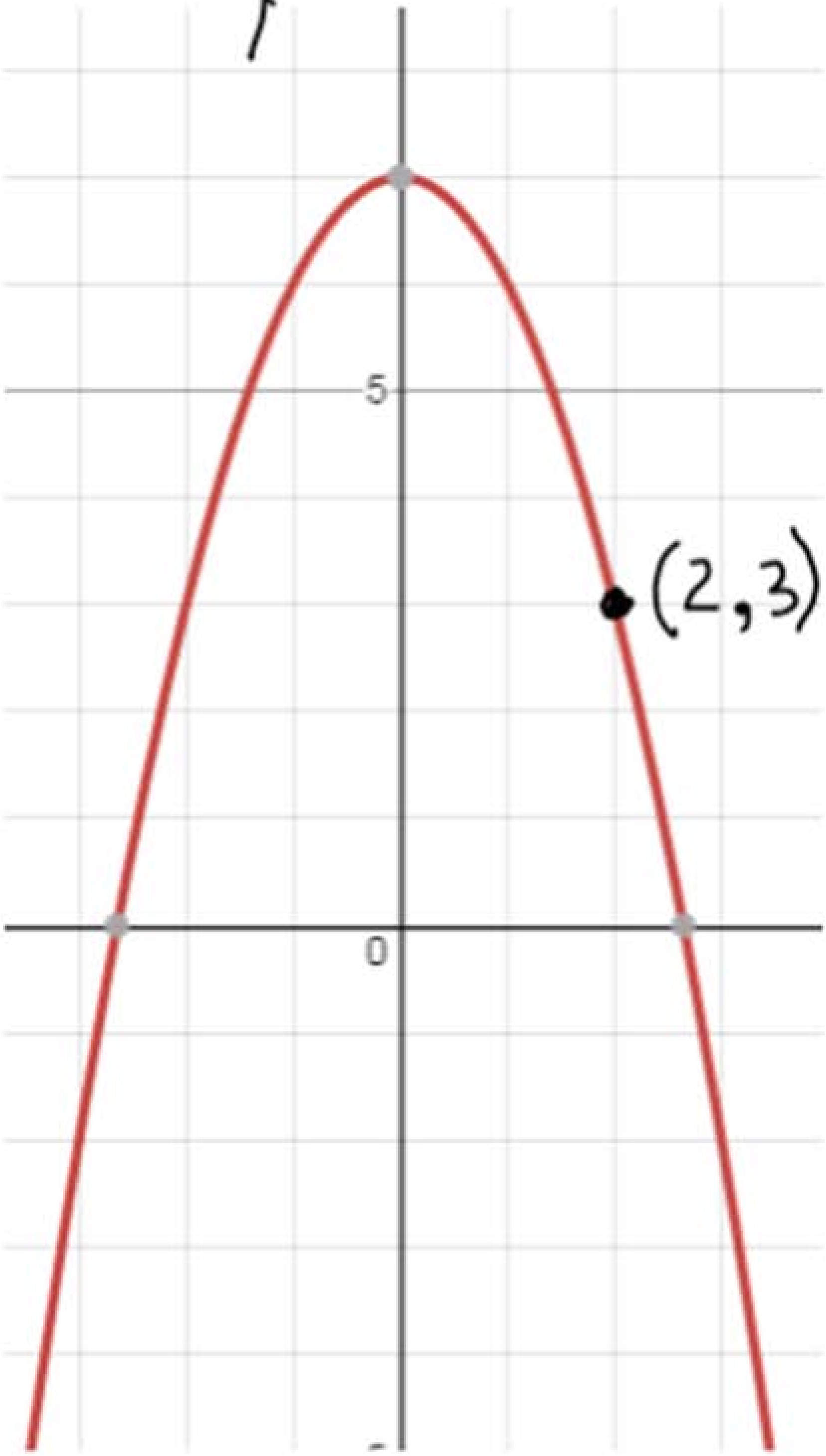
In either case, as Q approaches P along the curve, h approaches zero, $h \to 0$, and the secant line slope $4+h$ approaches 4.

#8) $y(x) = 7 - x^2$ $\quad P(2,3)$

$x_1 = 2$ $\qquad y_1(2) = 3$

$x_2 = 2 + h$ $\qquad y_2(2+h) = 7 - (2+h)^2$

$y = 7 - x^2$



Let m_s = secant line slope

$$m_s = \text{secant line slope} = \frac{y_2(2+h) - y_1(2)}{x_2 - x_1}$$

$$= \frac{7 - (2+h)^2 - 3}{(2+h) - 2}$$

$$= \frac{7-(4+4h+h^2)-3}{h}$$

$$= \frac{7-4-4h-h^2-3}{h}$$

$$= \frac{3 - 4h - h^2 - 3}{h}$$

$$= \frac{-4h - h^2}{h}$$

$$= \frac{h(-4-h)}{h}$$

$$= -4-h$$

we want the slope of the tangent line to the curve at $P(2,3)$. Let $h \to 0$.

m_t = slope of the tangent line.

$$m_t \to -4 \quad \text{as} \quad h \to 0$$

#12) $y(x) = 2 - x^3$ $P(1,1)$

$x_1 = 1$ $y_1(1) = 1$

$x_2 = 1$ $y_2(1+h) = 2 - (1+h)^3$

Let m_s = secant line slope

$$m_s = \text{secant line slope} = \frac{y_2(1+h) - y_1(1)}{x_2 - x_1}$$

$$= \frac{2-(1+h)^3-1}{(1+h)-1}$$

We will use Pascal's Triangle to expand $(1+h)^3$.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ \end{array}$$

$$\boxed{1 \quad 3 \quad 3 \quad 1} \longrightarrow a^3+3a^2b+3ab^2+b^3$$

$$(1+h)^3 = 1+3h+3h^2+h^3$$

$$m_s = \frac{2-(1+3h+3h^2+h^3)-1}{(1+h)-1}$$

$$m_s = \frac{2 - 1 - 3h - 3h^2 - h^3 - 1}{h}$$

$$m_s = \frac{1 - 3h - 3h^2 - h^3 - 1}{h}$$

$$m_s = \frac{-3h - 3h^2 - h^3}{h}$$

$$m_s = \frac{-h(3 + 3h - h^2)}{h}$$

$$m_s = 3 + 3h - h^2$$

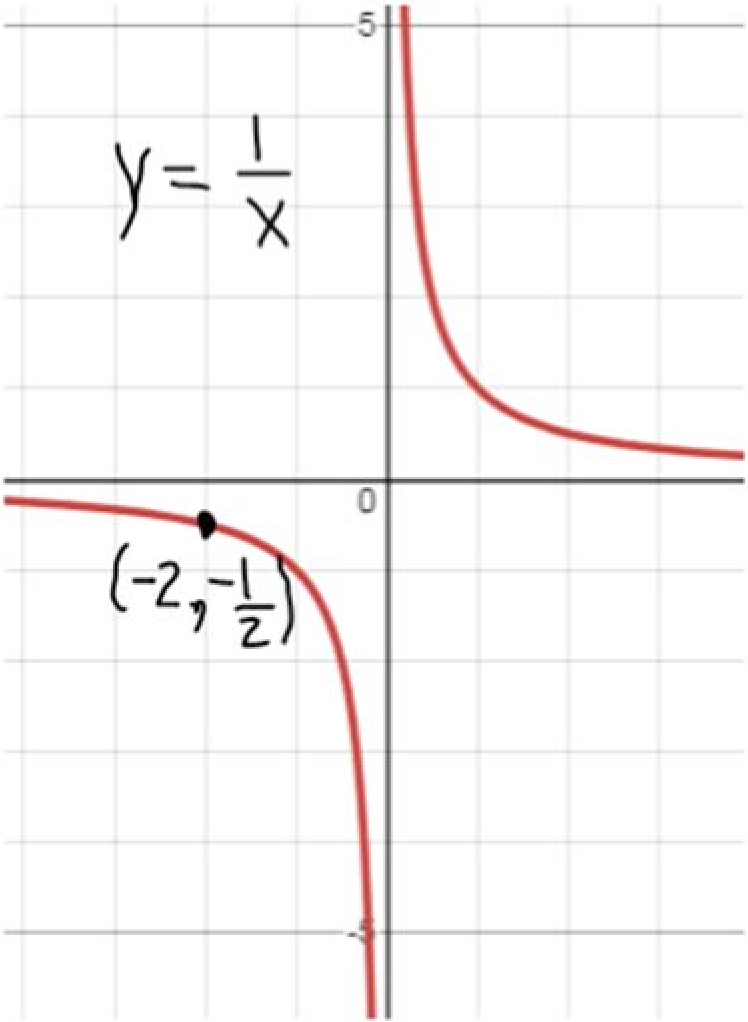
To determine the slope of the tangent line to the curve at $P(1,1)$, let $h \to 0$

m_t = slope of the tangent line.

$$m_t \to 3 \text{ as } h \to 0$$

#15)

$$y = \frac{1}{x} \qquad P\left(-2, -\frac{1}{2}\right)$$



$$x_1 = -2 \quad , \quad y_1(-2) = \frac{-1}{2}$$

$$x_2 = -2+h \qquad y_2(-2+h) = \frac{1}{-2+h}$$

$$m_s = \frac{y_2(-2+h) - y_1(-2)}{x_2 - x_1}$$

$$m_s = \frac{\frac{1}{-2+h} - \left(-\frac{1}{2}\right)}{(-2+h) - (-2)}$$

$$m_s = \frac{\left[\frac{1}{-2+h} + \frac{1}{2}\right]}{-2+h+2}$$

$$m = \frac{\left[\dfrac{2+(-2+h)}{2(-2+h)}\right]}{h}$$

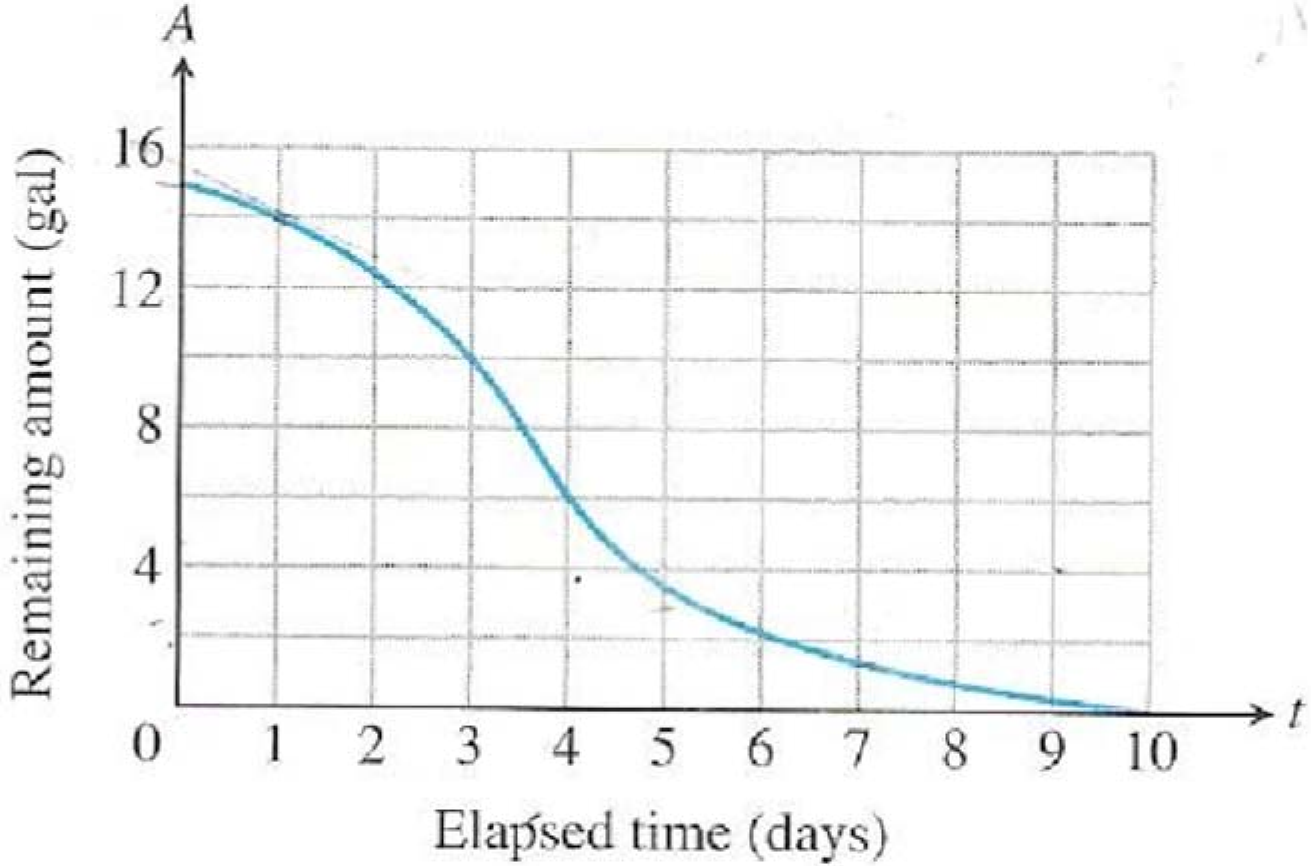
$$m_s = \left[\frac{h}{2(-2+h)}\right]\left(\frac{1}{h}\right)$$

$$m_s = \frac{1}{2(-2+h)}$$

To determine the slope of the tangent line to the curve at $P\left(-2, -\frac{1}{2}\right)$, let $h \to 0$

$$m_t \longrightarrow -\frac{1}{4} \quad \text{as} \quad h \to 0$$

#26)


A
16
12
8
4
0
Remaining amount (gal)
1
2
3
4
5
6
7
8
9
10
t
Elapsed time (days)

a

Elapsed time (days)

a.

Estimated Coordinates

$(0,15) \rightarrow (3,10)$

$$ARC = \frac{10-15}{3-0} = -\frac{5}{3} = -1.66 \ \frac{gal}{day}$$

$(0,15) \rightarrow (5,4.7)$

$$ARC = \frac{4.7-15}{5-0} = \frac{-10.3}{5} = -2.06 \, \frac{\text{gal}}{\text{day}}$$

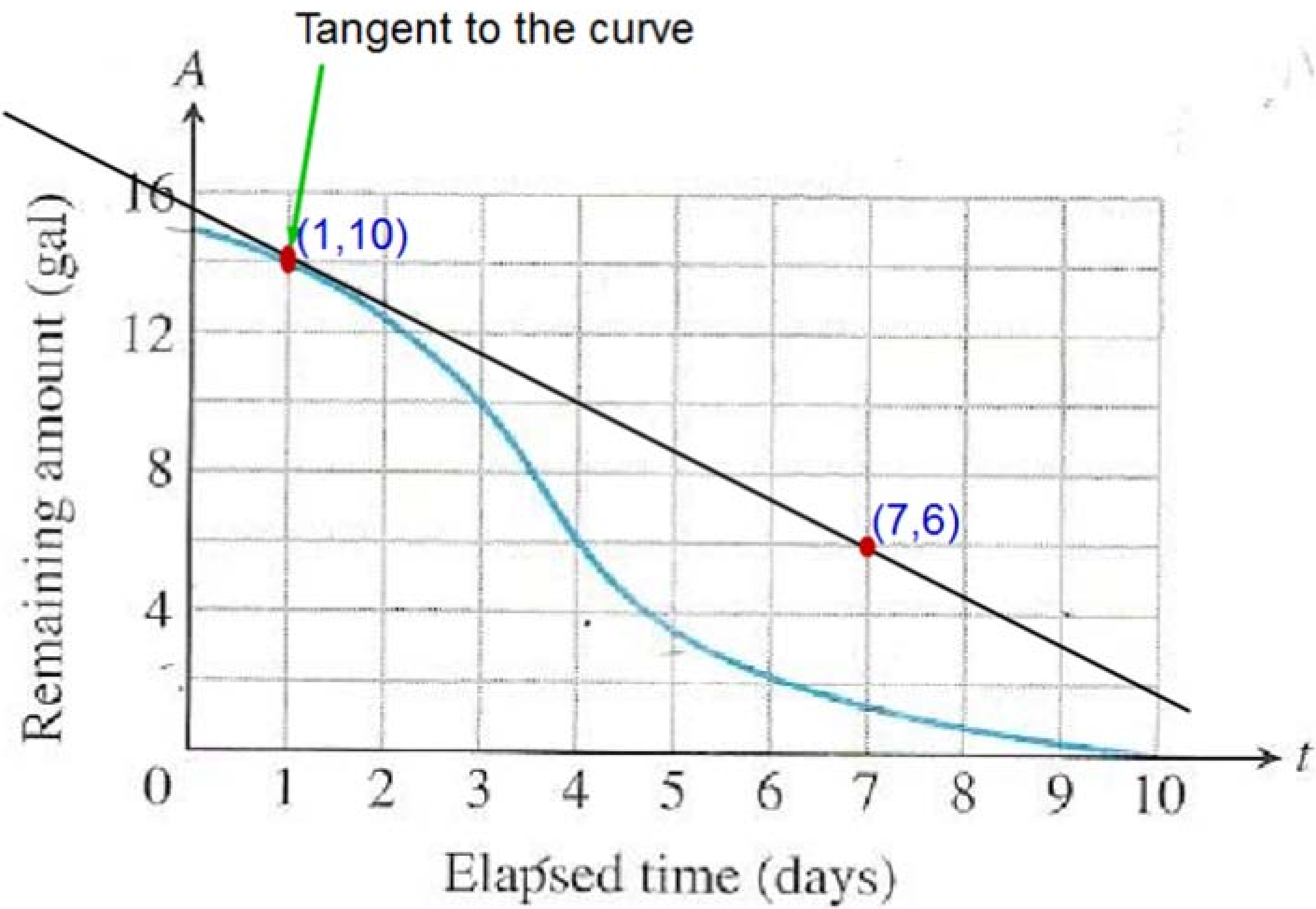
$(7,1) \longrightarrow (10,0)$

$$ARC = \frac{0-1}{10-7} = -\frac{1}{3} = -0.3\overline{3} \, \frac{\text{gal}}{\text{day}}$$

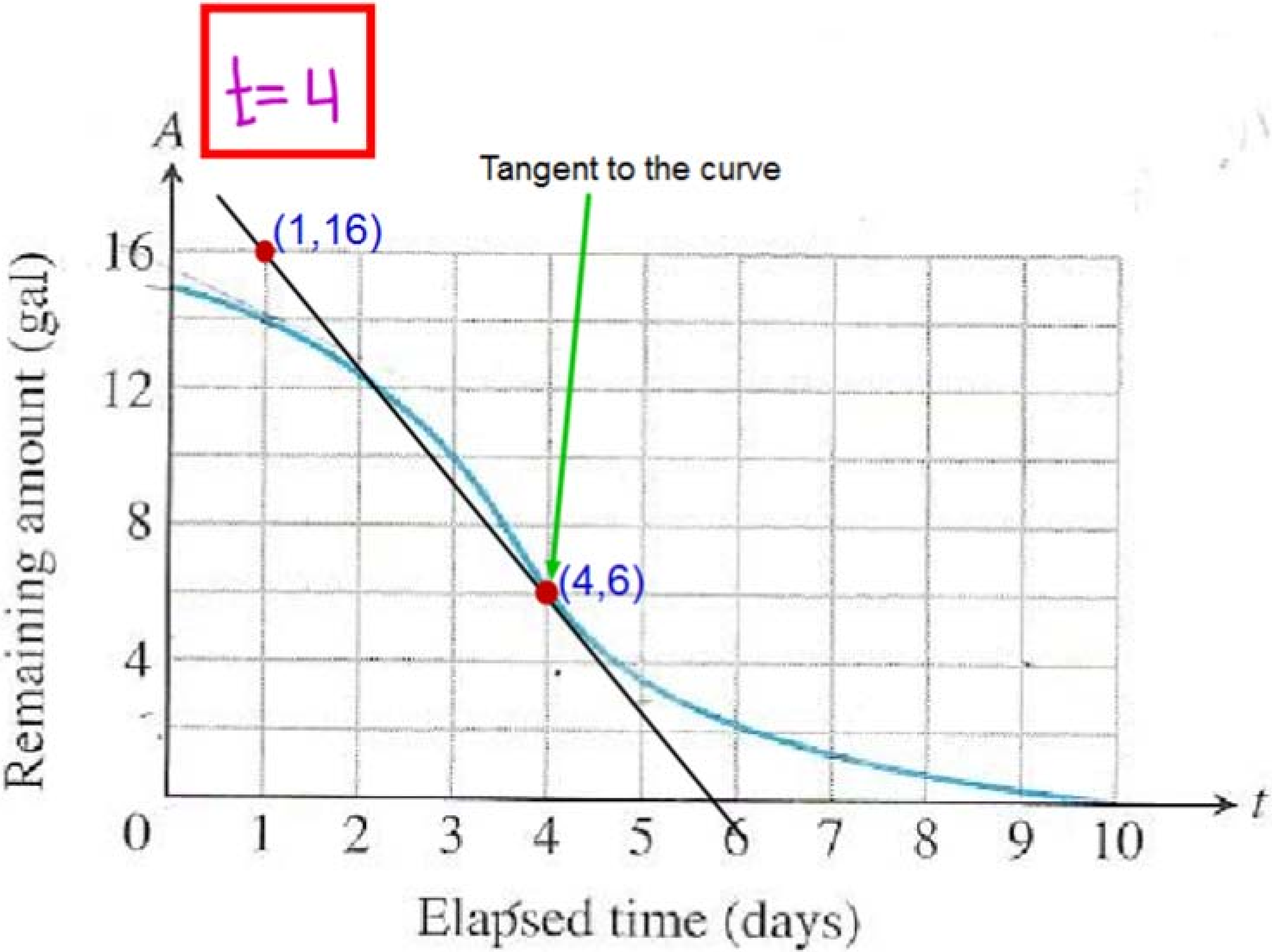
b. Estimate the instantaneous rate of gasoline consumptions at the times t=1, t=4 and t=8

Let IRC= instantaneous rate of consumption

t=1



$$IRC = \frac{6-10}{7-1} = \frac{-4}{6} = -\frac{2}{3} = -0.6\bar{6}\ \frac{gal}{day}$$

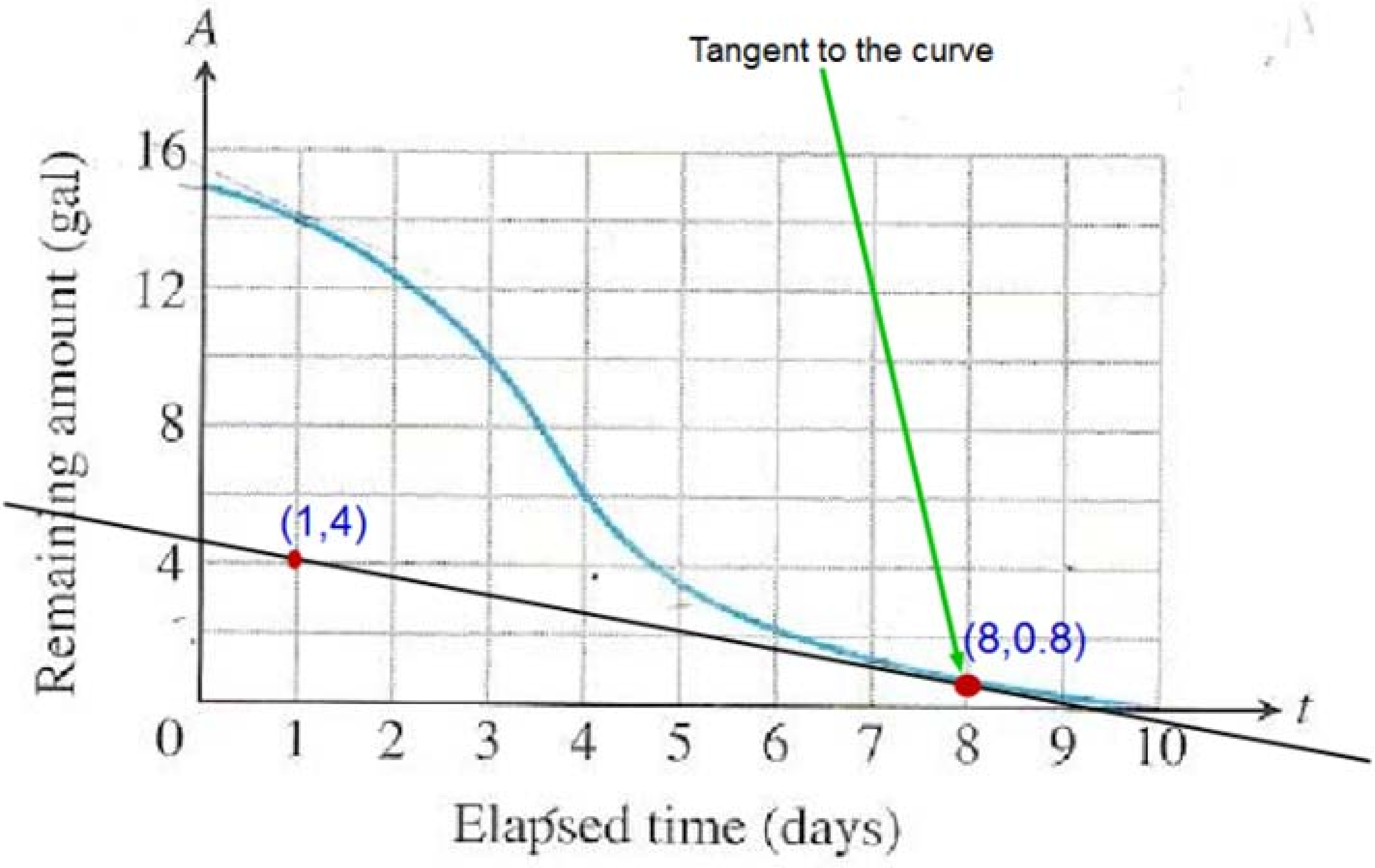


t=4
A
Tangent to the curve
(1,16)
(4,6)
16
12
8
4
0
1
2
3
4
5
6
7
8
9
10
t
Remaining amount (gal)
Elapsed time (days)

$$IRC = \frac{6-16}{4-1} = \frac{-10}{3} = -3\tfrac{1}{3} = -3.33\ \frac{gal}{day}$$

#26) $\boxed{t=8}$

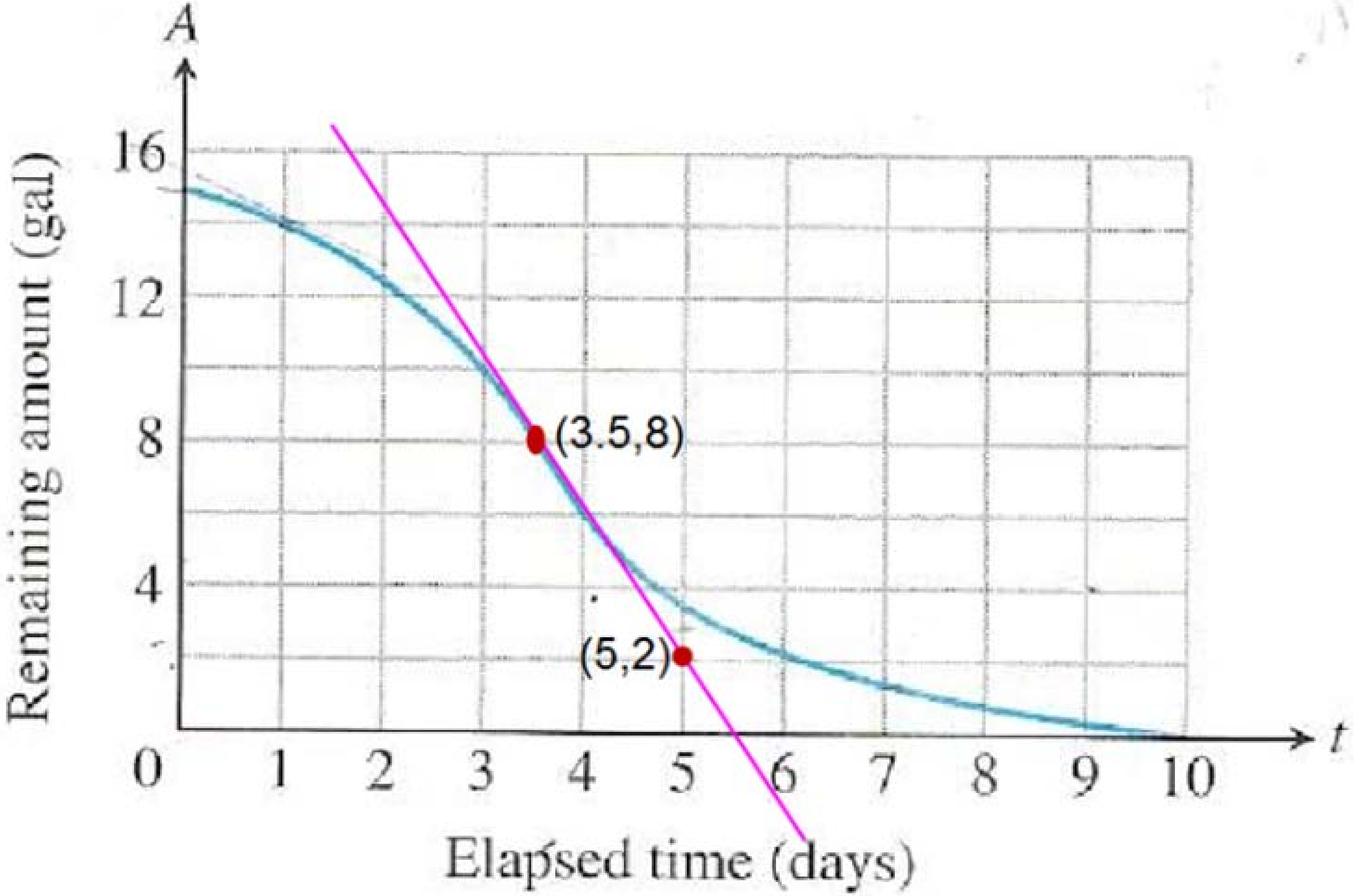
b)


A
Tangent to the curve
16
12
8
4
(1,4)
(8,0.8)
0
1
2
3
4
5
6
7
8
9
10
t
Remaining amount (gal)
Elapsed time (days)

$$IRC = \frac{0.8-4}{8-1} = \frac{-3.2}{7} = -0.457\,\frac{gal}{day}$$

C. Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

This problem is asking for the instantaneous rate of change at a specific time.



This is an estimate. The slope appears to be steepest at this point of tangency.

$$IRC = \frac{2-8}{5-3.5} = \frac{-6}{1.5} = -4\,\frac{gal}{day}$$

$$t = 3.5\,days$$